Analytically and Accurately Determined Quasi-Static Parameters of Coupled Microstrip Lines

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Abstract— **Using modified conformal mapping technique and magnetic-wall approximation, closed-form expressions for quasi-static parameters of coupled microstrip lines are determined accurately in this paper. They are found to be very accurate when compared with the well-recognized numerical solutions in the literature. Specifically, the present effective permittivities for both modes are accurate to within 0.4%, the even-mode characteristic impedance is accurate to within 1.2%, and the odd-mode impedance is accurate to within 1.8% for $w/h \geq 0.4$ and 3.8% for $0.4 > w/h \geq 0.1$. They are believed to be the most accurate closed-form formulas for coupled microstrip lines and should find applications in microstrip computer-aided design. In addition, two sets of existing equations have been checked against exact values or accurate results. According to the comparisons, they are either unacceptable or partially acceptable. Three data tables instead of figures are given for clarity.**

I. Introduction

PARALELL COUPLED microstrip lines are widely used in directional couplers, filters, and delay lines. Their quasi-static impedances and effective permittivities are the most important parameters in the analysis and design of the related components. Due to the inhomogeneity in these coupled lines, no exact solutions exist. As a result, much work has been done in determining these parameters either accurately by numerical methods [1], [2] or approximately by analytical methods [3]–[7]. Numerical studies can provide very accurate (exact in the sense of effective values) results, but they are not convenient in actual uses, especially in optimized design. On the other hand, analytical studies do generate closed-form expressions quite suitable for analysis and design, but the accuracies depend critically on the involved approximations. For this reason, analytical solutions with acceptable accuracies are always attempted if possible.

Today, various design equations for coupled microstrip lines are available in the literature. Accuracy is the problem. Through superposition of partial capacitances, Schwarzmann [3] has presented semiempirical formulas which are very approximate. Shamanna *et al.* [4] have designed nomograms which provide rough accuracy and are not appropriate for CAD uses. Akhtarzad *et al.* [5] have proposed a simple procedure for both synthesis and analysis of coupled microstrips, using the expressions for a single microstrip line as an intermediate. The original accuracy is around 10%. After corrections by Osmani [6], the error is still 6%. Garg and Bahl [7] have also given semiempirical equations for a limited geometrical range. The accuracies of the effective permittivities and even-mode characteristic impedance for an alumina substrate were found to be within 3% and the odd-mode impedance is sometimes in error of 8% [8]. In 1980, Hammerstad and Jensen [9] reported simple formulas, obtained using functional approximation to accurate models or even to numerical data, for both a single microstrip line and coupled microstrip lines. Their formulas for a single microstrip have been intensively tested and are very accurate up to 0.2%. In fact, Hammerstad's earlier models for a single microstripline [10] is already accurate to within 1%. However, their formulas for coupled microstrip lines have been checked only to some extent [11]. Based on Hammerstad and Jensen's results, Kirschning and Jansen [12] have remodeled some static parameters of the coupled lines, which have been claimed to be more accurate, to predict frequency-dependent properties by closed-form expressions. If these formulas had been widely proven to be accurate enough,

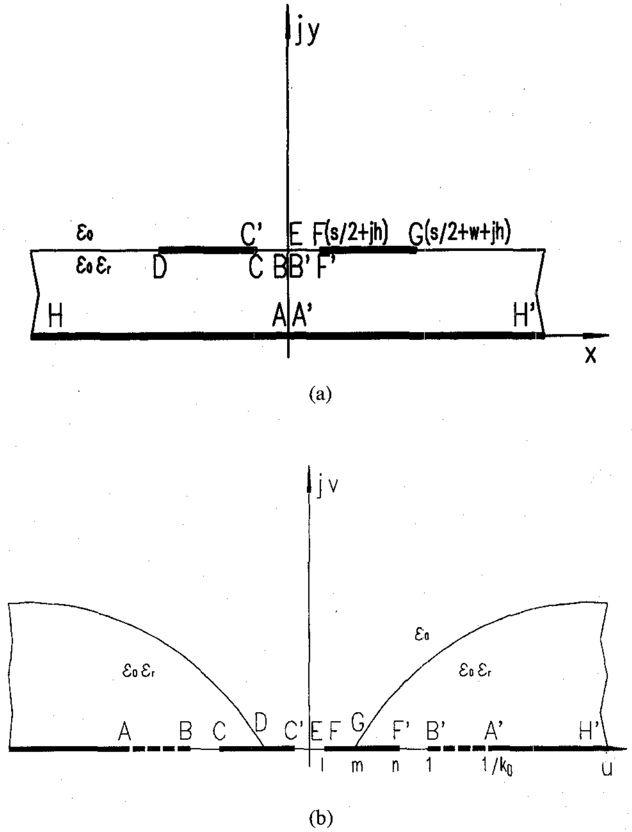

Fig. 1. Geometry of coupled microstrip lines in (a) z-plane and (b) t-plane.

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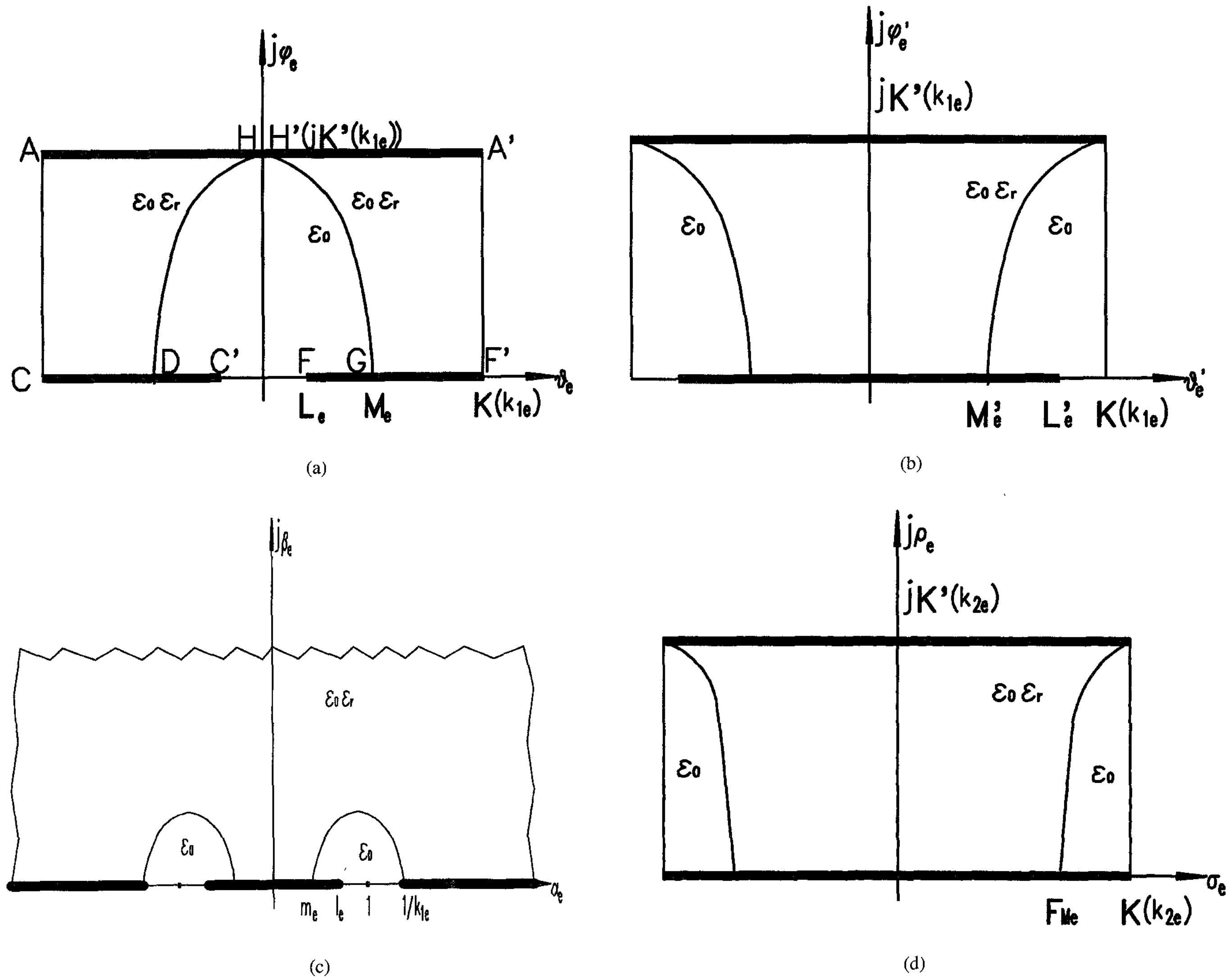

Fig. 2. Even-mode structure in (a) ξ_e-plane; (b) ξ_e'-plane; (c) η_e-plane; and (d) ζ_e-plane.

they would certainly have been useful in designing coupled microstrip lines.

The present work is to accurately analyze coupled microstrip lines by conformal mapping technique in conjunction with equivalent air-dielectric boundary approximation. This analysis has the following advantages: 1) The characteristic impedances and effective permittivities are expressed in terms of geometrical dimensions and permittivity in closed form. 2) The equivalent air-dielectric boundary approximation originally proposed by Wheeler [13] in the analysis of a single microstrip line has been proven to be excellent and his results are accurate up to 1%. Detailed conformal mapping analysis of coupled microstrip lines and closed-form expressions are given in the second section. The third section contains several tables other than figures to clearly compare various results. Conclusions and discussions are included in the last section.

II. Conformal Mapping Analysis

Fig. 1(a) shows the geometry of coupled microstrip lines and the chosen coordinate system. The structure consists of two identical signal strips of width w with a separation s on a dielectric substrate of thickness h and permittivity $\varepsilon_0\varepsilon_r$ on an infinite ground plane. The signal strips are assumed to be infinitely thin and long, and all the conductors are perfectly conducting. Under the above assumptions, the structure supports two normal quasi-TEM modes, i.e., even mode and odd mode. The following Schwarz–Christoffel transformation

$$z = C_1 \int_0^t \frac{t^2 - m^2}{\sqrt{(t^2-1)(t^2-k_0^{-2})}}\, dt + jh \qquad (1)$$

will map the interior of polygonal region H-A-B-C-D-C'-E-F-G-F'- B'-A'-H' in z-plane into the upper half of t-plane, where the corresponding points in t-plane are labeled, $0 < l < m < n < 1 < 1/k_0$, C_1 is a complex constant to be determined. According to the correspondence of some critical points in z-plane and t-plane, we readily obtain

$$m = \frac{1}{k_0}\sqrt{1 - \frac{E(k_0)}{K(k_0)}} \qquad (2)$$

$$\frac{\pi s}{4h} = K(k_0)E(\sin^{-1} l, k_0) - E(k_0)F(\sin^{-1} l, k_0) \quad (3)$$

$$\frac{\pi}{2h}\left(\frac{s}{2}+w\right) = K(k_0)E(\sin^{-1} m, k_0) - E(k_0)F(\sin^{-1} m, k_0) \quad (4)$$

$$\frac{\pi s}{4h} = K(k_0)E(\sin^{-1} n, k_0) - E(k_0)F(\sin^{-1} n, k_0) \quad (5)$$

where $\boldsymbol{K}(k_0)$ $[\boldsymbol{F}(x, k_0)]$ and $\boldsymbol{E}(k_0)$ $[\boldsymbol{E}(x, k_0)]$ are, respectively, complete (incomplete) elliptic integral of the first and second kind, with modulus k_0.

It is time for us to make analyses separately for even- and odd-mode case, but before doing that, we assume the slot between two strips as magnetic wall.

A. Even-Mode Case

In the case of even-mode excitation, the upper half of y-axis in z-plane is a magnetic wall. Consequently, the upper half of v-axis, A-B-C, and F′-B′-A′ in t-plane are also magnetic walls. Using the elliptic sinusoidal transformation

$$t = n\, sn(\xi_e, k_{1e}) \quad (6)$$

with $k_{1e} = nk_0$, we obtain the equivalent even-mode structure in ξ_e-plane as shown in Fig. 2(a). Because the upper half of φ_e-axis in ξ_e-plane is a magnetic wall, two symmetrical halves of the structure can be rearranged to form an exactly equivalent structure in ξ_e'-plane and

$$M_e' = K(k_{1e}) - F\left(\sin^{-1}\frac{m}{n}, k_{1e}\right) \quad (7)$$

$$L_e' = K(k_{1e}) - F\left(\sin^{-1}\frac{l}{n}, k_{1e}\right). \quad (8)$$

Of course, the capacitance of the symmetrical half of the new structure is our goal. Another elliptic sinusoidal transformation

$$\eta_e = sn(\xi_e', k_{1e}) \quad (9)$$

maps the rectangular region in ξ_e'-plane back to the upper half of η_e-plane with two signal strips contacted as shown in Fig. 2(c) and

$$m_e = sn(M_e', k_{1e}) \quad (10)$$

$$l_e = sn(L_e', k_{1e}). \quad (11)$$

A parallel-plate capacitor is obtained by the third elliptic sinusoidal mapping

$$\eta_e = l_e sn(\zeta_e, k_{2e}) \quad (12)$$

with $k_{2e} = l_e k_{1e}$ and

$$F_{Me} = F\left(\sin^{-1}\frac{m_e}{l_e}, k_{2e}\right). \quad (13)$$

As in the analysis of a single microstrip line by Wheeler [13], the air-dielectric boundary here is also an elliptical-looking curve. The next step is to assume the boundary to be a quarter of an ellipse and to use equivalent boundary approximation to get the final result as

$$\Delta x_e = \frac{(4-\pi)(4+\pi\varepsilon_r)}{4(4-\pi+2\pi\varepsilon_r)}[K(k_{2e}) - F_{Me}]. \quad (14)$$

The dielectric filling factor is given by

$$q_e = \frac{F_{Me} + \Delta x_e}{K(k_{2e})} \quad (15)$$

and the even-mode effective permittivity takes the usual form

$$\varepsilon_{re} = 1 + q_e(\varepsilon_r - 1). \quad (16)$$

Finally, the even-mode characteristic impedance is calculated by

$$Z_e = \frac{120\pi}{\sqrt{\varepsilon_{re}}}\frac{K'(k_{2e})}{K(k_{2e})} \quad (17)$$

where $\boldsymbol{K}'(k_{2e})$ is also the complete elliptic integral of the first kind, but with the complementary modulus $\sqrt{1-k_{2e}^2}$.

TABLE I
COMPARISON OF IMPEDANCE AND EFFECTIVE PERMITTIVITY DATA FOR $\varepsilon_r = 10.0$ AND $s/h = 0.2$

Dim.	Bryant & Weiss [1]		This method		Hammerstad & Jensen [9]	
w/h	Z_e	ε_{re}	Z_e	ε_{re}	Z_e	ε_{re}
0.05			172.98	6.18	170.70	6.44
0.10	152.98	6.25	151.02	6.25	149.46	6.47
0.30	111.08	6.54	110.83	6.51	109.53	6.67
0.50	90.40	6.77	90.37	6.73	89.48	6.86
0.70	76.88	6.96	76.89	6.93	76.35	7.02
1.00	63.18	7.21	63.18	7.19	62.94	7.24
1.30	53.80	7.41	53.78	7.41	53.67	7.43
1.60	46.93	7.59	46.90	7.59	46.85	7.60
2.00	40.17	7.78	40.13	7.79	40.10	7.78
w/h	Z_o	ε_{ro}	Z_o	ε_{ro}	Z_o	ε_{ro}
0.05			76.04	5.52	184.37	5.52
0.10	64.92	5.50	62.41	5.53	161.70	5.52
0.30	45.97	5.53	44.98	5.55	119.56	5.55
0.50	39.15	5.57	38.50	5.58	56.87	5.59
0.70	35.09	5.62	34.61	5.62	36.97	5.63
1.00	31.00	5.70	30.61	5.68	30.63	5.71

TABLE II
COMPARISON OF IMPEDANCE DATA FOR $\varepsilon_r = 1.0$ AND $s/h = 0.6$

Dim.	This method		Hammerstad & Jensen [9]		Kirschning & Jansen [12]	
w/h	Z_e	Z_o	Z_e	Z_o	Z_e	Z_o
0.05	376.23	233.98	374.04	237.59	374.09	236.99
0.10	330.69	196.20	329.52	196.74	329.26	196.09
0.20	282.22	161.09	281.05	161.01	280.66	160.45
0.30	252.15	142.15	250.85	142.18	250.48	141.71
0.40	230.05	129.49	228.73	129.60	228.42	129.21
0.50	212.55	120.09	211.27	120.24	211.03	119.92
0.60	198.11	112.68	196.90	112.83	196.72	112.56
0.70	185.86	106.58	184.73	106.70	184.61	106.48
0.80	175.27	101.41	174.22	101.51	174.15	101.32
0.90	165.98	96.93	165.00	97.00	164.97	96.85
1.00	157.74	92.99	156.83	93.03	156.83	92.91
1.10	150.36	89.49	149.50	89.48	149.54	89.39
1.20	143.71	86.30	142.90	86.29	142.97	86.22
1.30	137.67	83.45	136.90	83.39	136.99	83.34
1.40	132.16	80.82	131.42	80.73	131.53	80.70
1.50	127.11	78.41	126.39	78.28	126.52	78.26
1.60	122.45	76.02	121.76	76.02	121.90	76.01

B. Odd-Mode Case

In the case of odd-mode excitation, the upper half of y-axis in z-plane is now an electric wall. So, the upper half of v-axis, A-B, and B$'$-A$'$ in t-plane are also electric walls. The same transformation (6) in even-mode case is made, but with k_{1e} replaced by $k_{1o} = n$, to map the upper half of t-plane into a rectangular region. After completing further transformations, we have the same formulas as (7), (8), but with the subscript "e" replaced by "o" for current odd-mode case. Furthermore, (9)–(13) hold for the odd-mode case except that $k_{2o} = l_o$ if the above replacement is made. In addition

$$X = F(\sin^{-1} k_{1o}, k_{2o}). \tag{18}$$

In a similar way in the even-mode case

$$\Delta x_o = \frac{(4-\pi)(4+\pi\varepsilon_r)}{4(4-\pi+2\pi\varepsilon_r)}(X - F_{Mo}). \tag{19}$$

The remaining expressions for q_o, ε_{ro}, and Z_o are the same as (15)–(17), respectively, with the replacement of subscript "e" by "o."

III. IMPEDANCE AND EFFECTIVE PERMITTIVITY DATA

The expressions derived in last section have been programmed to calculate the characteristic impedances and effective permittivities for given dimensions and dielectric substrate in both cases. For the purpose of comparison, Hammerstad and Jensen's formulas [9] and their modified version by Kirschning and Jansen [12] have been also included in the program. The first check is to compare three kinds of results with well-recognized data by Bryant and Weiss [1] for $\varepsilon_r = 10.0$. Table I gives the comparison.

The following details are found in Table I. For the even mode, our results agree very well with Bryant and Weiss's accurate ones, with the error of the impedance less than 1.2% and that of effective permittivity, 0.15%. For the odd mode, our impedances are accurate to within 1.8% for $w/h \geq 0.4$ and 3.8% for $0.4 > w/h \geq 0.1$ while our effective permittivities are accurate to within 0.4%. This means that the proposed model is quite accurate. However, Hammerstad and Jensen's odd-mode impedances are acceptable (5% error) only for $w/h \geq 0.7$ and go wrong for $w/h < 0.7$ though their even-mode impedances and effective permittivities are accurate to

TABLE III
COMPARISON OF IMPEDANCE AND EFFECTIVE PERMITTIVITY DATA FOR $\varepsilon_r = 2.35$ AND $s/h = 1.0$

Dim.	This method				Hammerstad & Jensen [9]	
w/h	Z_e	ε_{re}	Z_o	ε_{ro}	Z_e	Z_o
0.05	262.30	1.82	198.62	1.69	255.07	200.35
0.10	228.83	1.83	168.17	1.70	224.14	168.43
0.25	182.43	1.87	129.86	1.70	179.65	129.73
0.50	145.33	1.91	102.94	1.72	143.43	102.87
0.75	123.18	1.94	88.10	1.73	121.69	88.01
1.00	107.62	1.97	77.98	1.75	106.37	77.85
1.25	95.85	1.99	70.39	1.76	94.75	70.22
1.75	78.97	2.03	59.43	1.79	78.09	59.25
2.25	67.34	2.06	51.84	1.82	66.59	51.56

within 2.3% and 3.3%, respectively, and odd-mode effective permittivities, 0.4%. As for Kirschning and Jansen's results, probably, there are still some misprints in their publication [12] in addition to the corrections because both even-mode and odd-mode impedances differ significantly from correct values except that their odd-mode effective permittivities are acceptable.

An interesting comparison is made in Table II for $\varepsilon_r = 1.0$ because the impedance in this case is important in using the effective dielectric constant concept.

It is seen that both Hammerstad's and Kirschning's results agree well with the accurate values in this paper in the case of $\varepsilon_r = 1.0$ and $s/h = 0.6$. The discrepancy is within 0.7% except for $w/h = 0.05$.

Further comparison of our results with Hammerstad and Jensen's ones for low dielectric constant is given in Table III.

Again, Hammerstad and Jensen's results are in good agreement with our data. Also, the data in the last row of Table III agree with Kirschning and Jansen's numerical solution [12]. More detailed comparison of present models with Hammerstad and Jensen's has been made for $w/h = 0.05$–3.0, $s/h = 0.1$–3.0, and $\varepsilon_r = 1.8$–18.0. The results show that Hammerstad and Jensen's functional approximate formulas are fairly accurate for a medium strip separation s (comparable to the substrate thickness h, i.e., $s/h \approx 1.0$), less accurate for a large $s\,(s/h > 3.0)$, and totally in error for the odd-mode impedance but accurate for the other three parameters for a small $s\,(s/h < 0.4)$. As a result of this, their formulas are not recommended for use with a small or large strip separation.

IV. CONCLUSION

Parallel coupled microstrip lines have been analyzed by modified conformal mapping technique together with a magnetic-wall approximation. This analysis yields analytical expressions for the quasi-static parameters, i.e., even- and odd-mode characteristic impedances and effective permittivities. After comparison with the well-recognized results generated by numerical methods, they have been found to be very accurate. Our effective permittivities for both modes are nearly exact, with maximum error 0.4%, and even-mode characteristic impedances are accurate to within 1.2%. Our odd-mode impedances are accurate to within 1.8% for $w/h \geq 0.4$ and 3.8% for $0.4 > w/h \geq 0.1$. To the author's knowledge, these results are the most accurate analytical ones. Through detailed comparisons, Hammerstad and Jensen's formulas do not satisfy the original accuracy specifications in the prescribed range of geometrical dimensions and permittivity. They can be either accurate for a medium strip separation or less accurate for a large strip separation, even totally incorrect for a small strip separation. Maybe, there are some misprints in Kirschning and Jansen's formulas. Why does the functional approximation in modeling coupled microstrip lines fail while it is success in modeling a single microstrip line? The following two reasons will give the answer: 1) There are no analytical and accurate solutions to these coupled lines which can be reduced to, for a given accuracy, simple expressions serving as a core for functional approximations, while their main characteristics are retained. 2) One more geometrical parameter is involved in modeling the coupled counterpart as compared with a single microstrip line. However, with the present analytical and accurate formulas for coupled microstrip lines, functional approximations to them will probably generate simple and accurate expressions.

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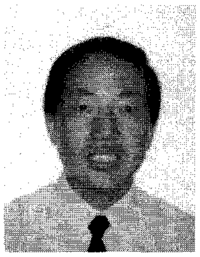

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